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Knowledge-Closure and Inferential Knowledge[1]

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Closure is the principle that a person, who knows a proposition p and knows that p entails q, also knows q. Closure is usually regarded as expressing the commonplace assumption that persons can increase their knowledge through inference from propositions they already know. In this paper, I will not discuss whether closure as a general principle is true. The aim of this paper is to explore the various relations between closure and knowledge through inference. I will show that closure can hold for two propositions p and q for numerous different reasons. The standard reason that S knows q through inference from p, if S knows p and knows that p entails q, is only one of them. Therefore, the relations between closure and inferential knowledge are more complex than one might suspect.

Keywords: closure, epistemology, inference, knowledge

Introduction

Knowledge-closure or *closure* is the principle that a person, who knows a proposition p and knows that p entails q, also knows q. Hence, knowing p and knowing that p entails q is a sufficient reason for knowing q. If closure holds, then knowledge is said to be closed under known entailment. Closure is usually regarded as expressing the commonplace assumption that persons can increase their knowledge through *inference* from propositions they already know. Closure as a general principle is treated controversially and numerous versions of closure are attacked and defended.[2] In

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[2] For an attack on closure see Dretske (1970 and 2005) and Nozick (1981). For a defence see Hawthorne (2005). For discussions of various versions of closure see Blome-Tillman (2006) or David and Warfield (2008).

this paper, I will not discuss whether closure as a general principle is true or not. The aim of this paper is to explore the various relations between closure and knowledge through inference. I will show that closure can hold for two propositions p and q for various reasons. Therefore, the relation between closure and inferential knowledge is less straightforward than one might suspect.

In this paper, I will proceed in the following way: I will firstly define different types of reasons for closure. I will secondly present examples for different reasons for closure. I will thirdly conclude that the relations between closure and inferential knowledge are rather complex. I will fourthly show that the same holds for modified versions of closure as well.

1. *Closure*

Closure is often formulated as the principle that a person knows a proposition q, if she knows a proposition p and knows that p entails q. This formulation might conceal the fact that closure is usually regarded as a general principle, valid for all persons and all propositions. In order to make its general character and its necessity more explicit, one can formulate closure as following:

Closure:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$

This formulation makes explicit that closure holds with necessity for all persons w and all propositions x and y.

Any logical implication of the form $\Box(a \rightarrow b)$ is true iff *in every possible world* the material implication $(a \rightarrow b)$ is true. Therefore, closure is true iff $((K(p) \wedge K(p \text{ entails } q)) \rightarrow K(q))$ is true for all persons and all propositions in all possible worlds.

Since closure is a general statement with respect to persons and propositions, one can restrict its domain in each of these aspects.[3] I will call versions of closure with restricted domains "particular closure". If closure is restricted to a single person S, it has the following structure:

Closure$_{(S)}$:

- $\Box\forall x\forall y((K_S(x) \wedge K_S(x \text{ entails } y)) \rightarrow K_S(y))$

This restricted version indicates that closure holds for a particular person S and all propositions x and y. Closure$_{(S)}$ is true iff $((K(p) \wedge K(p \text{ entails } q)) \rightarrow K(q))$ is true for person S and all propositions in all pos-

[3] Not only generality, but also necessity is a characteristic feature of closure. Therefore, one can constrain closure by modifying its modality as well. The contingent version of closure has the following structure: $\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. This version of closure states that a material implication holds for all persons and all propositions. Therefore, it is weaker than the common version of closure, which claims that a logical implication holds.

sible worlds. Restricting closure can be useful for different reasons: On the one hand, one can imagine notoriously irrational persons who fail to reason in accordance with the closure principle. On the other hand, one can define an ideally rational person as somebody who reasons in accordance with closure. Considering such cases, it can be useful to restrict closure to a certain class of persons who fulfil minimal conditions of rationality or to restrict it to a single person S.

One can also restrict closure to particular propositions respectively to ordered pairs of propositions. A closure principle, which is restricted to particular propositions p, q has the following structure:

Closure$_{(p,\,q)}$:

- $\Box\forall w((K_w(p) \wedge K_w(\text{p entails q})) \rightarrow K_w(q))$

Closure$_{(p,\,q)}$ is true iff $((K(p) \wedge K(\text{p entails q})) \rightarrow K(q))$ is true in all possible worlds for all persons and the pair of propositions p, q.

We usually understand closure as a principle holding for all propositions and all persons. This view is supported by the fact that a standard strategy to attack closure is finding pairs of propositions p, q such that closure does not hold. Dretske (2005) for example argues that closure is false because it does not hold, if q is a heavyweight proposition like "There are material objects", but p is not a heavy weighty proposition like "There is a computer in front of me." This argumentation line against closure is only available, if closure is a general principle, which is not restricted to a certain class of propositions.

Obviously, one can easily restrict closure to a particular person S *and* a particular pair of propositions p, q in the following way:

Closure$_{(S,\,p,\,q)}$:

- $\Box((K_S(p) \wedge K_S(\text{p entails q})) \rightarrow K_S(q))$

Closure$_{(S,\,p,\,q)}$ is true iff $((K(p) \wedge K(\text{p entails q})) \rightarrow K(q))$ is true in all possible worlds for person S and the pair of propositions p, q.

To sum up, closure is a general principle for all persons and all propositions, which can easily be restricted to particular persons and/or particular propositions.

2. *Inferential Knowledge*

We usually regard closure as somehow expressing the idea that persons can gain knowledge through inference. Here the question arises: What is inferential knowledge? It is a commonplace assumption, even among philosophers, that persons can increase their knowledge trough inference from propositions they already know. However, theories about how persons can acquire new knowledge through inference can vary in different aspects. In the following, I will give a short overview of various theories about inferential knowledge.

Inference is a kind of relation between a class of propositions, which function as premises, and a proposition, which functions as the inference's conclusion. One type of theories about inferential knowledge concerns the truth-preserving relations between premises and conclusion. The paradigmatic case for knowledge through inference is knowledge by deduction. If there is knowledge through inference, a fact, which is regarded as indisputable, then there is knowledge through deductive inference. The question, which other types of inference can provide a source for inferential knowledge is more controversial. Inductive inferences are one plausible candidate. Further candidates are abductive inferences such as inferences to the best explanation. In cases of abductive inference, the direction of inference and the direction of truth preservation are opposite: The truth of p is sufficient for the truth of q, but the direction of inference is from q to p.[4]

A further theory type about inferential knowledge concerns the epistemic status of the premises. It seems indisputable that justified premises are a necessary condition for inferential knowledge, but we are inclined to accept that the premises not only need to be justified; they also have to be known. Furthermore, different views about the psychological impact of inference can be taken. One can argue either that inference is merely a process of justification or that inference is also a process of belief acquisition. In the latter case, it is plausible to assume that the process of inference has to *cause* the belief for being inferential knowledge. In the first case, persons can know a proposition through inference, which they already believe for other reasons. This is not possible if the belief acquisition itself is an aspect of inferential knowledge.

One can also take different views towards the question whether knowledge about the truth preserving relation between premises and conclusion is necessary for inferential knowledge or not. On the one hand, demanding that such knowledge about the truth preserving relation is necessary is in danger of leading into an infinite regress. On the other hand, it seems controversial whether persons can acquire inferential knowledge without having any knowledge about the truth preserving relation at all.

Each theory about single aspects of inferential knowledge such as the truth preserving relation between premises and conclusion, the epistemic status towards the premises, the causal relations between inferring and believing or knowledge about the truth preserving relation between premises and conclusion imply theories about *necessary* condition for inferential knowledge. One possible way to obtain theories about *sufficient* conditions is by combining these theories about different single aspects. One can, for example, hold the view that S knows q through inference from p iff S knows p, p entails q, S draws an inference from p to q, and S knows that p entails q. This formulation already indicates how different views about necessary and sufficient

[4] The structure of abductive inference will become clearer later.

conditions for inferential knowledge can lead to different views about closure. These connections will become clearer in the last chapter.

3. *Inferential Knowledge and Naïve Closure*

If closure is based on the assumption that inference is an potential knowledge source, what are exactly the relations between inferential knowledge and closure? Williamson (2000) for example claims, that intuitive closure is the principle that knowing $p_1, \ldots, p_n$, competently deducing q and thereby coming to believe q is in general a way of coming to know q. Contemporary discussions about the validity of closure give us another hint towards the expected relations between closure and inferential knowledge. There are, generally speaking, three possible attitudes towards closure, firstly accepting it, secondly refuting it for general reason or thirdly criticising it for technical reasons and trying to modify it. This third path is probably the most popular one. Technical objections to closure share the following structure: They point out that there are propositions p and q for persons S such that the antecedent of closure is true for S, p and q, but S fails to know q, because S fails to know q *through inference* from p and (p entails q). Therefore, closure is false, because it fails to be true for p and q.

This line of argumentation against closure shows that closure can only be true, if the conditions formulated in the antecedent are, to a certain extend, sufficient for knowing q *through inference* from p and (p entails q). One might suggest that closure is, therefore, the principle that knowing p and knowing that p entails q are together sufficient conditions for knowing q *through inference* from p and (p entails q). Taking this naïve view on closure means to defend the following thesis:

Naïve Closure:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow (K_w(y)$ *through inference* from x))

Naïve closure states that knowing p and knowing that p entails q is sufficient for knowing q through inference from p and (p entails q). It excludes cases of knowing p, knowing that p entails q but knowing q for other reasons than through inference from these two propositions. Closure and naïve closure share the same antecedent, but the consequent of closure is simply knowing q, whereas the consequent of naïve closure is knowing q *through inference* from p. Therefore, naïve closure is stronger than closure.

Versions of naïve closure can vary in different aspects. One can for example assume that q is known through inference from p *and* (p entails q) or that knowing that (p entails q) only plays an implicit role in inference and needs not to be involved in the process of inference at all. I leave it open whether inferring q from p also involves an inference from (p entails q) or not. Therefore, I will use the notion of naïve closure in a broader sense.

In the following, I will present various counterexamples against the view that naïve closure is true. I will argue that this view on the relation between closure and inferential knowledge is false and that it has to be replaced by a more complex picture. I will leave open whether closure is a valid principle or not, but I will argue that no version of closure holds, *because* its corresponding naïve closure holds. To be fair, this naïve view is not explicitly defended in literature. Hawthorne clearly diagnosis that this view is false by noting that closure "allows that there be a knowable P and an entailed Q such that deductive inference from P to Q was not a possible route to knowing Q". (2005: 41) But there is, to my best knowledge, no systematic investigation of the relation between closure and inferential knowledge at hand yet.

I will proceed in the following way: I will firstly define different types of closure. I will secondly show that closure can hold for particular propositions p and q for different reasons. I will present various examples, which are instances of closure, but not of naïve closure. I will conclude that naïve closure is false. Hence, the relation between closure and knowledge through inference is more complex than one might suspect.

4. *Types of Reasons for Closure*

For investigating possible reasons for closure, it is useful to categorize them in a systematic way: A *sufficient reason for closure* is by definition a fact such that if it obtains, then closure is true. This can either be the fact that propositions have a specific property, that propositions stand in a specific relation to each other, that persons have a specific property, that persons and propositions stand in a specific relation to each other, or something else. Since other facts than properties of propositions or persons or relations between them are no plausible candidates for sufficient reasons for closure, I will leave this possibility aside. In the following, I will distinguish three types of sufficient reasons for closure:

Definition: Type-1-reason

- In all possible worlds, there is one and the same property or relation a sufficient reason for closure, which is a property of all propositions or a property of all persons, a relation between all propositions or a relation between all propositions and persons.

Definition: Type-2-reason

- There is for every proposition or person o*ne property in all possible worlds*, or there is for every pair of propositions one relation in all possible worlds, or there is for every triple of two propositions and one person a relation in all possible worlds and this proposition or relation is a sufficient reason for closure.

Definition: Type-3-reason

- For every proposition or person, there is *in every possible world a property*, or for every pair of propositions, there is *in every possible world one relation*, or for every triple of two propositions and a person, there is *in every possible world one relation*, which is a sufficient reason for closure for this proposition or this person or this pair of propositions or this triple of two propositions and one person.

The structural differences between these three types of reasons become clearer, if we have a closer look at their formal structure.

A type-1-reason for closure can either be a property of the premise, a property of the conclusion, a relation between propositions or a relation between propositions and persons. Using second-order-logic for quantifying over properties and relations and using x, y, .. as variables for propositions, w as a variable for persons and X, Y, … as variables for properties and relations, we can state that there is a type-1-reason for closure iff one of the following theses is true:

1. $\exists X(\forall x(Xx) \wedge \Box\forall w\forall x\forall y(Xx \rightarrow (\textit{closure})))$
2. $\exists X(\forall y(Xy) \wedge \Box\forall w\forall x\forall y(Xy \rightarrow (\textit{closure})))$
3. $\exists X(\forall x\forall y(Xxy) \wedge \Box\forall w\forall x\forall y(Xxy \rightarrow (\textit{closure})))$
4. $\exists X(\forall w(Xw) \wedge \Box\forall w\forall x\forall y(Xw \rightarrow (\textit{closure})))$
5. $\exists X(\forall w\forall x\forall y(Xwyx) \wedge \Box\forall w\forall x\forall y(Xwxy \rightarrow (\textit{closure})))$

Here, *closure* is an abbreviation for the proposition "$\Box((K_w(x) \wedge K_w(\text{x entails y})) \rightarrow K_w(y))$".

In the first case, the sufficient reason is a property of all premises, in the second case it is a property of all conclusions, in the third case it is a relation between all propositions, in the fourth case it is a property of all persons and in the last case it is a relation between all persons and properties.

There is a type-2-reason for closure iff the following disjunction is true:

- $\forall w\forall x\forall y(\Box\exists X((Xx) \wedge \Box(Xx \rightarrow (\text{closure})) \vee \Box\exists X((Xy) \wedge \Box(Xy \rightarrow (\text{closure})) \vee \Box\exists X((Xxy) \wedge \Box(Xxy \rightarrow (\text{closure})) \vee \Box\exists X((Xw) \wedge \Box(Xw \rightarrow (\text{closure})) \vee \Box\exists X((Xwyx) \wedge \Box(Xwxy \rightarrow (\text{closure}))$

Type-1-reasons for closure and type-2-reasons differ concerning the scope of quantifiers. In case of type-1-reasons, there is one property or one relation for all propositions or persons and in case of type-2-reasons there is for all propositions or persons a property or relation, which implies closure.

The definition of type-2-reasons does not imply that there is exactly one reason for every proposition or person. Hence, closure can hold for two propositions p and q for two or more reason. This will become more obvious later.

Type-2-reasons exclude the possibility that there are propositions p and q such that there are in different possible worlds different reasons for closure for p, q. Type-3-reasons allow this possibility.

There is a type-3-reason for closure iff the following disjunction is true:

- $\forall w \forall x \forall y \Box(\exists X((Xx) \wedge \Box(Xx \rightarrow (\text{closure})) \vee \exists X((Xy) \wedge \Box(Xy \rightarrow (\text{closure})) \vee \exists X((Xxy) \wedge \Box(Xxy \rightarrow (\text{closure})) \vee \exists X((Xw) \wedge \Box(Xw \rightarrow (\text{closure})) \vee \exists X((Xwyx) \wedge \Box(Xwxy \rightarrow (\text{closure}))$

Type-2-reasons and type-3-reasons differ concerning the scope of the necessity operator. In case of type-2-reasons there is a property or relation, which implies closure for p and q. If there is a type-3-reason for closure, then there is for all propositions in every possible world a property or relation, which implies closure, but it need not to be one and the same property or relation in different possible worlds.

The truth conditions for type-1-reasons are stronger than the ones for type-2-reasons, which are stronger than the ones for type-3-reasons: If there is a type-1-reason for closure, then there is a type-2-reason and a type-3-reason for closure as well. If there is a type-2-reason for closure, then there is also a type-3-reason for it.

Types of Reasons for Particular Closure

So far, I have presented different types of reasons for the general version of closure, with all propositions and all persons as its domain. But one can easily restrict closure to particular propositions, to a particular person or to particular propositions *and* a particular person. I will now present and discuss different types of reasons for these versions of particular closure. There are three types of possible reasons for general closure. In contrast to this, there are the following *two* types of reason for closure for a particular pair of propositions p, q:

Type-2-reason for closure$_{(p,\,q)}$:

- There is *one property of p or one property of q or one relation between p and q*, which is *in all possible worlds* a sufficient reason for closure$_{(p,\,q)}$.

Type-3-reason for closure$_{(p,\,q)}$:

- There is *in every possible world a property of p or a property of q or a relation between p and q*, which is a sufficient reason for closure$_{(p,\,q)}$.

In the first case, one and same property or relation is the reason for closure$_{(p,\,q)}$ in all possible worlds, in the second case the reason can vary in different possible worlds. It will become clearer later that there are certain implication relations between type-2-reasons for closure and type-2-reasons for closure$_{(p,\,q)}$ on the one hand and between their type-3-

reasons on the other hand. For terminological reasons, there is, therefore, no type-1-reason for closure$_{(p,\ q)}$.

The formal structures of types of reasons for closure$_{(p,\ q)}$ are similar to the ones for general closure. In the following, $closure_{(p,\ q)}$ is an abbreviation for "$\forall w((K_w(p) \wedge K_w(\text{p entails q})) \rightarrow K_w(q))$".

There is a type-2-reason for closure$_{(p,\ q)}$ iff one of the following theses is true:

1. $\exists X((Xp) \wedge (Xp \rightarrow \Box closure_{(p,\ q)}))$
2. $\exists X((Xq) \wedge (Xq \rightarrow \Box closure_{(p,\ q)}))$
3. $\exists X((Xpq) \wedge (Xpq \rightarrow \Box closure_{(p,\ q)}))$
4. $\exists X((Xwpq) \wedge (Xwpq \rightarrow \Box closure_{(p,\ q)}))$

There is a type-3-reason for closure$_{(p,\ q)}$ iff the following disjunction is true:

- $\Box((\exists X(Xp) \wedge (Xp \rightarrow closure_{(p,\ q)})) \vee (\exists X(Xq) \wedge (Xq \rightarrow closure_{(p,\ q)})) \vee (\exists X(Xpq) \wedge (Xpq \rightarrow closure_{(p,\ q)})) \vee (\exists X(Xwpq) \wedge closure_{(p,\ q)})))$

The truth condition for type-2-reasons is stronger than the ones for type-3-reasons: If there is a type-2-reason for closure$_{(p,\ q)}$, then there is a type-3-reason as well.

One can formulate type-2-reasons and type-3-reasons for a particular closure for p, q and a particular person S analogously by adding a property of the person S or a relation between p, q and S as further possible reasons.

Implication relations between reasons for closure:

I have defined three types of reasons for general closure and two types for particular closure$_{(p,\ q)}$. The following implication relations between reasons for general closure and reasons for particular closure hold:

1. There is a type-1-reason for closure iff for every pair of proposition p, q there is the *same* type-2-reason for particular closure$_{(p,\ q)}$.
2. There is a type-2-reason for closure iff for every pair of proposition p, q there is a type-2-reason for particular closure$_{(p,\ q)}$.
3. There is a type-3-reason for closure iff for every pair of propositions p, q there is a type-3-reason for particular closure$_{(p,\ q)}$.

These relations imply:

- If there are pairs of propositions p, q and r, s such that there are different type-2-reason for closure$_{(p,\ q)}$ and closure$_{(r,\ s)}$, then there is no type-1-reason for closure.
- If there are propositions p and q such that there is only a type-3-reason for closure$_{(p,\ q)}$, then there is no type-1-reason and no type-2-reason for closure.

In the following, I will show that there are different type-2-reasons for different particular closures and that there are particular closures which only hold for type-3-reasons. I will conclude that, if closure holds, then only for a type-3-reason.

5. *Reasons for Closure*

For making my argumentation more explicit, I will firstly present possible examples for type-1-reasons for closure. Most of them are trivially false.

5.1 *Possible type-1-reasons for closure*

A type-1-reason for closure is one and the same reason for all propositions and all persons. Such a reason can either be a property for all propositions, a relation between all propositions or a property for all persons or a relation between all persons and all propositions.

Type-1-reason 1: All propositions are false

If all propositions are false, then no proposition can be known and, therefore, closure holds for all propositions. The implication chain is the following: If $\Box\forall x(Fx)$, then $\Box\forall w\forall x(\neg K_w(x))$, then $\Box\forall w\forall x\forall y\neg(K_w(x) \wedge K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. However, not all propositions are false. Therefore, this possible reason for closure does not obtain.

Type-1-reason 2: Every proposition is necessarily known

If every proposition is necessarily known, then the following implication holds: If $\Box\forall w\forall y(K_w(y))$, then $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. However, not all propositions are true. Since knowledge implies truth, not all propositions can be known. Therefore, this possible sufficient reason for closure is also not the case.

In the first case, a property of the *entailing* proposition would be the reason for closure, in the second case one of the *entailed* proposition would be.

Type-1-reason 3: Nobody can know any proposition

A property of a person that trivially implies closure is that nobody can know any proposition. The implication chain is the following: If $\Box\forall w\forall x(\neg K_w x)$, then $\Box\forall w\forall x\forall y\neg(K_w(x) \wedge K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. Assuming that no proposition can be known is implausible either.

Type-1-reason 4: No entailment relation between any two propositions

If there does not exist any entailment relation between two propositions, then the implication chain leading to closure is the following: If $\Box\forall x\forall y(\neg(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y(\neg K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y\neg(K_w(x) \wedge K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. Again, this possible sufficient reason does obviously not obtain. It is an hypothetical example for a relation between two propositions as a possible type-1-reason.

Type-1-reason 5: No known *entailment relation between any two propositions*

If nobody can know any entailment relation between any two propositions p and q, then closure is implied in the following way: If $\Box\forall w\forall x\forall y\neg(K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y\neg(K_w(x) \wedge K_w(x \text{ entails } y))$, then $\Box\forall w\forall x\forall y(((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$. This is an example for a type-1-reason, which is a relation between propositions *and* persons. It is as implausible as the others above.

The possible sufficient reasons for closure presented until now are the following: No proposition is true, all propositions are known, no proposition can be known, there is no entailment relation between any two propositions, no entailment relation can be known. All these possible reasons are obviously not fact. In contrast to this, one might have the intuition that the following possible type-1-reason for closure is true.

Type-1-reason 6: Naïve Closure

Naïve closure, which has already been introduced, is the view that if a person knows p and knows that p entails q, then she knows q through inference from p. According to naïve closure, there is a single reason why closure holds for all propositions and all persons: It is the fact that whenever a person knows a proposition p and knows that p entails q, then she has inferential knowledge of q.

Naïve closure and closure share the same premises, but the consequence of naïve closure, which is the thesis that q is known *through inference*, is stronger then the one of closure, which is only the thesis that q is known. Therefore, naïve closure implies closure. If naïve closure is true, then, hence, there exists a type-1-reason for closure. The implication chain from naïve closure to closure is simply: If $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y \text{ through inference } x))$, then $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y)) \rightarrow K_w(y))$.

Closure does not hold necessarily for exactly one reason. If, for example, $\Box\forall x(\neg Kx)$ would hold, then this would be a reason for closure and for naïve closure as well. Since naïve closure is a reason for closure as well, there would exist two reasons for closure: $\Box\forall x(\neg Kx)$ and naïve closure. I do not claim that the six examples for type-1-reasons presented above are a complete list of possible type-1-reasons.

5.2 *Type-2-reasons for closure*

There is a type-2-reason for *general* closure, if there is a type-2-reason for every *particular* closure for a pair of propositions. Therefore, I will present and discuss various type-2-reasons for *particular* closure first and then illustrate its consequences for general closure.

Type-2-reason 1 for closure$_{(p,\,q)}$: p is necessarily false

If p is necessarily false, then p is necessarily unknown and then closure necessarily holds for p and any other proposition q. In this case, the implication chain is the following: If $\Box\neg(p)$, then $(\Box\forall w(\neg K_w(p))$, then $\Box\forall w\neg(K_w(p) \wedge K_w(p \text{ entails } q))$, then $\Box\forall w\neg((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$. Here is an example: p is the proposition that 2+2=7. q is the proposition that Paris is the capital of France. Since p is necessarily false it is true that every person who knows that 2+2=7 and knows that Paris is the capital of France, if 2+2=7, also knows that Paris is the capital of France. The same holds for p and any other proposition like the false proposition that Paris is the capital of Argentina. This is an example for a closure-implying property of the *entailing* proposition.

Type-2-reason 2 for closure$_{(p,\,q)}$: p is necessarily unknown

A further example for a closure-implying property of the entailing proposition is that p is necessarily unknown. In this case, the implication chain is the following: If $(\Box\forall w(\neg K_w(p))$, then $\Box\forall w\neg(K_w(p) \wedge K_w(p \text{ entails } q))$, then $\Box\forall w\neg((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$. If, for example, p is the continuum hypothesis and q is again the proposition that Paris is the capital of France, then closure holds for p and q because nobody can know p.

Type-2-reason 3 for closure$_{(p,\,q)}$: p does not entail q

If p does not entail q, then this is necessarily so. In this case, it is necessarily unknown that p entails q and, therefore, closure$_{(p,\,q)}$ is true. The implication chain is here: If $\neg(p \text{ entails } q)$, then $\Box\neg(p \text{ entails } q)$, then $(\Box\forall w(\neg K_w(p \text{ entails } q))$, then $\Box\forall w\neg(K_w(p) \wedge K_w(p \text{ entails } q))$, then $\Box\forall w\neg((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$. If, for example, p is the proposition that 2+2=4 and q is the proposition that the earth is round, then closure holds for p and q because p does not entail q.

Type-2-reason 4 for closure$_{(p,\,q)}$:
It is necessarily unknown *that p entails q*

If it is necessarily unknown that p entail q, then closure holds for p, q. The implication chain here is: If $\Box\forall w\neg K_w(p \text{ entails } q)$, then $\Box\forall w\neg(K_w(p) \wedge K_w(p \text{ entails } q))$, then $\Box\forall w((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$. To be honest no example of an entailment relation which cannot be known comes to my mind. This is rather a hypothetical type-2-reason for particular closure.

Type-2-reason 5 for closure$_{(p, q)}$: q is necessarily known

If everybody knows proposition q necessarily, then particular closure for any proposition p and q is trivially true. The implication chain is short in this case: If $\Box\forall w(K_w(q))$, then $\Box\forall w\forall x((K_w(x) \wedge K_w(x \text{ entails } q)) \rightarrow K_w(q))$. It is subject of philosophical discussion, whether there exist propositions, which everybody necessarily knows. One candidate for such a proposition is "I am here now." Another one might be "1+1=2". I will not argue here for or against necessarily known propositions, the only point I want to stress is the following: *If* there are such propositions, then particular closure holds for these propositions and any other proposition. If q is, for example, the necessarily known proposition "I am here now.", then closure holds for any p and q, no matter whether p is true or false, whether it is known or unknown or whether it entails q or not. p can, therefore, be any of the following propositions: "The earth is round", "The earth is flat", "2+2=4". One can argue that not everybody necessarily knows propositions like "I am here now." or "1+1=2", but only those persons who understand the involved concepts. In that case every version of closure for q holds, which is restricted to those persons who understand the involved concepts.

Type-2-reason 6 for closure$_{(p, q)}$: If q is true, then q is necessarily known

Implications like the closure principle are transitive relations: If a implies b and b implies c, then a implies c. If there is, hence, an implication relation $\Box\forall w(\alpha \rightarrow K_w(q))$ holding for which $\Box\forall w\forall x((K_w(x) \wedge K_w(x \text{ entails } q)) \rightarrow \alpha$ holds as well, then closure is valid for q and every other proposition p and every person: If a proposition p is known, then p is true. If one knows that p entails q, then it is true that p entails q. Therefore, knowing p and knowing that p entails q, implies that q is true. If, therefore, q is known if q is true, then closure holds for q and every other proposition p. The implication chain is the following: If $\Box\forall w(q \rightarrow K_w(q))$, then $\Box\forall w((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$. The reason is that $\Box\forall w((K_w(p) \wedge K_w(p \text{ entails } q))$ implies q.

As in the case of necessarily known propositions, it is subject of discussion whether there are such propositions, which are necessarily known, if true. One candidate are propositions about current own mental states like the proposition "I am now having a blue-experience". If one accepts such a theory of omniscience concerning own mental states, then she accepts that the implication $\Box\forall w(q \rightarrow K_w(q))$ holds, if q is a proposition about current own mental states. If $\Box\forall w(q \rightarrow K_w(q))$ holds, then closure holds for q and any proposition p, no matter, whether p is true or false, whether it is known or unknown or whether it entails q or not. Examples for p are "The earth is round", "The earth is flat" or "2+2=4". Again, I will not argue here for or against omniscience concerning certain propositions. I only want to point out that *if* there are propositions, which are necessarily known if true, then particular closure holds for these propositions and any other proposition.

Type-2-reason 7 for closure$_{(p,\,q)}$*: If p and (p entails q) are known, then q is known* through inference *from p*

This type-2-reason 7 for closure$_{(p,\,q)}$ is the restricted analogy to naïve closure. Knowing q through inference from p is a stronger conclusion then just knowing q. Therefore, the thesis above is stronger then closure and, hence, a possible sufficient reason for it. The truth of the closure principle is closely connected to the view that we can extend our knowledge through deductive inference. Taking this connection into account, knowing q through inference from p can be regarded as the *standard reason* for particular closure for p and q. I will call any other reason for closure than the standard reason an *alternative reason*. Nevertheless, it is more difficult then one might expect to find propositions p, q for which the standard reason necessarily holds, which implies that it is impossible to know q for any alternative reason. One example is that p is the only axiom of a formal calculus and that q is one of its theorems, which can only be known through inference from p.

There are more examples at hand if the implication that knowing p and knowing that (p entails q) implies knowing q through inference from p is restricted to a single person. Here is an example: Robert is watching a soccer game between team A and team B. Robert knows that if A wins the match, then C wins the championship. There is no other way for Robert to know whether C will succeed. If p is the proposition that A defeats B, and q is the proposition that C wins the championship, then naïve closure holds for p, q and Robert: $\Box((K_R(p) \wedge K_R(p \text{ entails } q)) \rightarrow K_R(q \text{ through inference from } p))$ is true.

Type-2-reason 8 for closure$_{(p,\,q)}$*:* *p can only be known through inference from q*

If p can only be known through inference from q, then knowing q is a necessary condition for knowing p and, therefore, the implication $\Box\forall w(K_w(p) \rightarrow (K_w(q))$ holds. The implication relations of closure are monotone: if a implies b, then (a and b) implies c as well. If $\Box\forall w(K_w(p) \rightarrow (K_w(q))$ is true, then $\Box\forall w((K_w(p) \wedge \alpha) \rightarrow (K_w(q))$ is true for any additional premise α. Therefore, closure holds for p and q, if p can only be known through inference from q. The implication chain is simply: If $\Box\forall w(K_w(p) \rightarrow (K_w(q))$, then $\Box\forall w((K_w(p) \wedge K_w(p \text{ entails } q)) \rightarrow K_w(q))$.

The standard reason for closure occurs, if q is known through inference from p, if p and (p entails q) are known. In this standard case, the direction of inference is from p to q. However, if p can only be known through inference from q, then the direction of inference is the opposite one from q to p.

If p can only be known through inference from q, what about the inference from q to p? Here one can distinguish different cases concerning different aspects: The first parameter is the type of inference: It is a common view that deductive inference is one possible way of gaining knowledge, but usually deductive inferences are not the only types of inferences for acquiring inferential knowledge. Other candidates, which I will discuss here, are inductive inferences or abductive inferences respectively inferences to the best explanation.

The second parameter concerns the question whether p is known through inference from q alone or from q plus additional premises. I will hence, distinguish different cases of knowing p only through inference from q along the following two parameters:

1. Type of inference from q to p: deductive/inductive/abductive
2. Inference from q alone: yes/no

By combining these two parameters, we can acquire six different sub cases of knowing p only through inference from q. In the following, I will give examples for each of these sub cases. I will not argue that inductive or abductive inferences are valid sources of inferential knowledge. But if they are, then closure$_{(p,\, q)}$ can hold for the following reasons.

a) Deductive *inference from q alone*

Example: q is the only axiom of a calculus and p is a theorem, which can only be known through inference from q. In this case closure holds for p and q, but the reason is not the standard case that q is always known through inference from p, but that p can only be known through inference from q.

Usually, the possibilities of gaining knowledge vary from person to person. Therefore, it is much easier to find examples for reasons for closure, which are also restricted to particular persons. Here is an example: Jason knows that he and a person who is unknown to him are both applying for a job. He also knows that one of the two will receive a letter of acceptance and the other one a letter of rejection. Jason receives an envelope opens it, reads the letter and knows that he has been accepted. Jason knows through inference that the other applicant received the rejection. He also knows that he received the acceptance, if the other applicant received the rejection. However, Jason can only know that the other applicant received the rejection by inferring it from his knowledge that he received the acceptance.

We can analyze this example more systematically in following way:

p: The other applicant received a letter of rejection.
q: Jason received a letter of acceptance.

Jason can only know p through inference from q. Therefore, $\Box(K_J(p) \rightarrow K_J(q))$. Therefore, $\Box(K_J(p) \wedge K_J(p \rightarrow q)) \rightarrow K_J(q)$. However, Jason does not know q through inference from p and (p entails q). In contrast, p must be known through inference from q. The direction of Jason's inference is from q to p and not from p to q.

Closure$_{(p, q)}$ holds for Jason at least in all nearby possible worlds. But closure$_{(p, q)}$ needs not to hold for any other person for the same reason as for Jason. The secretary of the company, which hired Jason, has other possibilities to know who got the job than opening the envelope, as Jason has to do.

b) Deductive *inference from q plus other premises*

Example: q and r are axioms of a formal calculus and p is a theorem, which can only be known through inference from q and r.

Example for a particular person: Frank knows that April, May and June are female and the only persons in seminar room 9.5. Frank infers that everybody in room 9.5 is female.

- p: Everybody in room 9.5 is female.
- q: April is in room 9.5 and female.
- r: May is in room 9.5 and female.
- s: June is in room 9.5 and female.
- t: April, May and June are the only persons in room 9.5.

If Frank is in a situation that he can only know p through inference from knowing q, r, s and t, then $\Box(K_f(p) \rightarrow K_f(q))$ is true and, hence, $\Box(K_f(p) \wedge K_f(p \rightarrow q)) \rightarrow K_f(q)$ is true as well. However, Frank knows p through inference from q plus other propositions. He does not know q through inference from p as in the standard case.

There are, at least generally, for all persons the same possibilities for deductive reasoning in formal calculi. In case of *deductive* reasoning from q to p one can, therefore, find examples for closure$_{(p, q)}$ which hold for all persons. However, for inductive and abductive reasoning to the best explanation, the situation is different. The possibilities of inductive and abductive reasoning can vary from person to person. Therefore, I will present in the following examples for inductive and abductive reasoning for particular persons.

c) Inductive *inference from q alone*

Example:

- p: All swans are white.
- q: S_1 is swan and white, S_2 is swan and white… and S_n is a swan and white.

If Adelaide is in a position that she can only know that all swans are white by inductively inferring it from her empirical knowledge about the whiteness of single swans S_1, … S_n, then closure$_{(p, q)}$ holds for Adelaide, because of an inductive inference from q to p.

d) Inductive *inference from q plus other premises*

Again, Frank knows that April, May and June are female and in seminar room 9.5, but he does not know that nobody else is there. Frank infers by induction that everybody in room 9.5 is female.

- p: Everybody in room 9.5 is female.
- q: April is in room 9.5 and female.
- r: May is in room 9.5 and female.
- s: June is in room 9.5 and female.

If Frank is in a situation, that he can only know p through inference from knowing q, r and s, then $K_f(p) \rightarrow K_f(q)$ is true and, therefore, also $(K_f(p) \wedge K_f(p \rightarrow q)) \rightarrow K_f(q)$. Closure$_{(p, q)}$ is true for Frank, because he can only know p through *inductive* inference from q plus other premises.

e) Abductive *inference from q alone*

Example:

- p: It has been raining.
- q: The street is wet.

If Jodie is in a position that she can only know that it has been raining through abductive inference to the best explanation from her knowledge that the street is wet, then $K_j(p) \rightarrow K_j(q)$ is true and, therefore, $(K_j(p) \wedge K_j(p \rightarrow q)) \rightarrow K_j(q)$ is true.

f) Abductive *inference from q and other premises*

- p: It has been raining.
- q: The street is wet.
- r: The street has not been cleaned today.

If Peter can only know that it has been raining by inferring it from his knowledge that the street is wet and that the street has not been cleaned today, then, again, $K_p(p) \rightarrow K_p(q)$ and, therefore, $(K_p(p) \wedge K_p(p \rightarrow q)) \rightarrow K_p(q)$. p can only be known through abductive inference from q and other premises.

To sum up, for different pairs of propositions p, q (and particular persons S) there exist various type-2-reasons. However, a particular closure for a pair of propositions can hold for more than one reason. It is e.g. possible that $\Box\forall w((K_w(p) \wedge K_w(p \rightarrow q)) \rightarrow K_w(q))$ is true because q is necessarily known, if true *and* because p does not entail q. In a next step, I will present examples for type-3-reasons for particular closure.

5.3 *Type-3-Reasons for Closure*

There is a type-3-reason for closure$_{(p, q)}$, iff there is *in every possible world a property of p or q or a relation between p and q*, which is a sufficient reason for closure for p and q, but it need not to be same property or relation in every possible world. In contrast, closure for p, q can hold in different possible worlds for various different reasons. The same holds for closure$_{(S, p, q)}$, restricted to a particular person S. Furthermore, there is only a type-3-reason for general closure iff there are propositions p, q, such that there is only a type-3-reason for closure$_{(p, q)}$. Hence, I will present examples for *particular* closures, which hold for type-3-reasons first. I will, next conclude that there is a type-3-reason for general closure as well.

The necessity implication $\Box((K_s(p) \wedge K_s(p \text{ entails } q)) \rightarrow K_s(q))$ is true iff in every possible world the material implication $((K_s(p) \wedge K_s(p \text{ entails } q)) \rightarrow K_s(q))$ is true, which is the case iff in every possible world the antecedent is false or the consequent is true. Therefore, $\Box((K_s(p) \wedge K_s(p \text{ entails } q)) \rightarrow K_s(q))$ is true, iff in every possible world at least one of the following propositions is true:

- P1: $\neg K_s(p)$
- P2: $\neg K_s(p \text{ entails } q)$
- P3 : $K_s(q)$

Each of the propositions P1-P3 implies the contingent version of closure$_{(S, p, q)}$, which is $((K_s(p) \wedge K_s(p \text{ entails } q)) \rightarrow K_s(q))$. The propositions P1-P3 do not exclude each other. Therefore, the contingent version of closure$_{(S, p, q)}$ can be true for more than one reason.

If in different possible worlds different of the propositions P1-P3 are true, then there are in different possible world different reasons for closure$_{(S, p, q)}$. In these cases, there is a type-3-reason for closure$_{(S, p, q)}$. Here is an example:

Example 1:

p: Today is Thursday.
q: Tomorrow is Friday.
Reasons for closure$_{(F, p, q)}$ in different possible worlds w_1, w_2 ...:
w_1: Today is not Thursday.
w_2: Frank does not know that today is Thursday.
w_3: Frank does not know that if today is Thursday, then tomorrow is Friday.
w_4: Frank knows that tomorrow is Friday by reading a newspaper.
w_5: Frank knows that tomorrow is Friday by looking at his watch.
w_6: Frank knows that tomorrow is Friday by asking Mary.
w_7: Frank knows that tomorrow is Friday by inferring it from his knowledge that he will marry in two days, which is a Saturday.
w_8: Frank knows that today is Thursday and infers that tomorrow is Friday.
...

In w_1 and w_2 $\neg K_f(p)$ is true, in w_3 $\neg K_f(p \rightarrow q)$ is true and in $w_4 - w_8$, $K_f(q)$ is true, although for different reasons. The standard reason, which is the inference from p to q, obtains in w_8. If in every other possible world w_n one of the propositions P1-P3 is true, then closure holds for Frank and the two propositions "Today is Thursday" and "Tomorrow is Friday". Since $closure_{(F, p, q)}$ is true for different reasons in different possible worlds, it is true for a type-3-reason.

Example 2:

Mary is participating at a game show. She can choose between two doors. The show master tells Mary, that behind one door, there is a cabriolet as the first prize and behind the other door a basket of apples as the consolation prize. In fact, the cabriolet is behind door 1 and the basket of apples behind door 2. The two propositions p and q are:

p: There is a cabriolet behind door 1.
q: There is a basket of apples behind door 2.

Mary knows that p and q imply each other. If Mary can open either door 1 or door 2 and if there is no other reason for Mary to know, what there is behind the two doors, then there exist different reasons for $closure_{(M, p, q)}$ in different possible worlds:

> Case 1: Mary opens door 1, sees a cabriolet and infers that there is a basket of apples behind door 2.
> Case 2: Mary opens door 2, sees a basket of apples and infers that there is a cabriolet behind door1.

In case 1 $(K_m(p) \wedge K_m(\text{p entails q})) \rightarrow K_m(q))$ holds because $(K_m(p) \wedge K_m(\text{p entails q})) \rightarrow (K_m(q))$ through inference from p) holds: Whenever Mary knows p, then she knows q through inference from p. In case 2, the same closure holds because for the reason that $(K_m(p) \rightarrow K_m(q))$ is true, because Mary can only know p through inference from q.

If case 1 or case 2 occurs at least in all nearby possible worlds, then $closure_{(m, p, q)}$ is true. Since it holds for different reasons in different possible worlds, it again holds for a type-3-reason. Numerous other examples for restricted versions of closure, which hold for a type-3-reason, can be found.

There is a type-2-reason for $closure_{(p, q)}$, if p is necessarily false or unknown, if q is necessarily known or necessarily known if true or if p does not entail q. In these cases, $closure_{(p, q)}$ is somehow trivially true. However, these are the less interesting examples of closure. Those cases of closure, which are true for less trivial reasons, are of much more philosophical interest. However, if $closure_{(p, q)}$ is not trivially true, then there are not many possibilities left for type-2-reasons anymore: The two remaining possibilities are, firstly, the standard reason that q is always known through inference from p, if p and (p entails q) are known, and, secondly, that p can only be known through inference from q.

Propositions can typically be known for various different reasons: Propositions about the external world, for example, can be known through different empirical methods, by inference or by testimony.

Logical or mathematical propositions can usually be known through various processes of inferences. In all these cases, there exists a type-3-reason for closure for these propositions. The case that one can only know q through inference from p or that p can only be known through inference from q are rather exceptions than the usual case. If, therefore, $closure_{(p, q)}$ is true for non-trivial reasons, then typically for a type-3-reason and not for a type-2-reason.

6. *Conclusions*

So far, I have been investigating the reasons for general closure as well as for versions of particular closure. Now, I can summarize the gained results as following:

Conclusion1:

- There exist pairs of propositions p, q such that $closure_{(p, q)}$ is true for a type-2-reason.

Examples for such type-two-reasons for $closure_{(p, q)}$, which I have been presenting, are:

1. p is necessarily false
2. p does not entail q
3. It is impossible to know that p entails q
4. q is necessarily known
5. If q is true, then q is necessarily known
6. If p and (p entails q) are known, then q is known through inference from p
7. p can only be known through inference from q

There also exist triples of two propositions *and* a particular person S such that $closure_{(S, p, q)}$ is true for a type-2-reason as well.

I have also shown that particular closures for *different* pairs of propositions can hold for *different* type-2-reasons. The following conclusion makes this explicit:

Conclusion2:

- There exist pairs of propositions p, q and r, s such that $closure_{(p, q)}$ and $closure_{(r, s)}$ hold for *different* type-2-reasons.

One example is the following: $closure_{(p, q)}$ is true because p does not entail q and $closure_{(r, s)}$ is true because r can only be known through inference from s. The same holds for versions of closure, which are also restricted to a person S.

Furthermore, I have shown that particular closure for propositions p, q can hold for type-3-reasons, as the following conclusion claims:

Conclusion3:

- There exist pairs of propositions p, q such that $\text{closure}_{(p,\ q)}$ only holds for a type-3-reason.

The same holds for version of closure, which are also restricted to a person S.

I have defined different types of reasons for general closure as well as for particular closure. The implication relations between these reasons are the following:

- There is a type-1-reason for general closure iff for every pair of proposition x, y there is the *same* type-2-reason for $\text{closure}_{(x,\ y)}$.
- There is a type-2-reason for general closure iff for every pair of proposition x, y there is a type-2-reason for $\text{closure}_{(x,\ y)}$.
- There is a type-3-reason for general closure iff there is at least one pair of propositions x, y such that there is only a type-3-reason for $\text{closure}_{(x,\ y)}$.

Conclusion2 states that there exist pairs of propositions p, q and r, s such that $\text{closure}_{(p,\ q)}$ and $\text{closure}_{(r,\ s)}$ holds for different type-2-reasons. Conclusion3 is the claim that there exists a pair of propositions t, u such that $\text{closure}_{(t,\ u)}$ only holds for a type-3-reason. Because of the implication relations stated above, each of these two conclusions implies:

- There is no type-1-reason for closure.

Conclusion3 also implies:

- There is no type-2-reason for closure.

Therefore, it follows:

- If closure is true, then there is a type-3-reason for it.

As mentioned earlier, this paper does not aim to argue for or against closure as a general principle. Its purpose is rather to investigate possible reasons for closure in general and the role of inferential knowledge as one such reason especially. We can sum up the outcome of the investigations until now as following: I have defined three types of reasons for closure. I have shown that closure can hold for different propositions for various different reasons. I concluded that, if closure is true, then only for a type-3-reason. In a next step, I will investigate the consequences of this outcome for the relations between closure and inferential knowledge.

7. *Closure and Inferential Knowledge*

Conclusions for Naïve Closure

Naïve closure is the following general claim:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(\text{x entails y})) \rightarrow K_w(\text{y through inference x}))$

I have shown that closure can hold for particular propositions for various different reasons. If p is necessarily false, if p does not entail q or if it is impossible to know that p entails q, then naïve closure for p, q holds for trivial reasons. But if q is necessarily known or necessarily known if true or if p can only be known through inference from q, then naïve closure does not hold for these propositions. Neither general naïve closure nor any other type-1-reason for general closure is fact.

But the fact that *general* naïve closure is false does not imply that any *particular* naïve closure for propositions p, q and persons S is false too. There are propositions p, q such that particular naïve closure holds for p, q and any person. In these cases naïve closure$_{(p,\ q)}$ is true. Furthermore, there are propositions p, q and particular persons S such that particular naïve closure holds for p, q and S. In these cases naïve closure is the type-2-reason for closure$_{(p,\ q)}$ respectively for closure$_{(S,\ p,\ q)}$.

There exist propositions p, q such that particular closure holds for a type-3-reason. In these cases, there are in different possible worlds different reasons for the contingent version of closure$_{(p,\ q)}$. In some possible worlds it might be the case that q is known through inference from p, if p and (p entails q) are known. In these possible worlds the standard reason for closure$_{(p,\ q)}$ is fact. In these cases the following contingent variants of naïve closure is true in possible worlds w_n:

- $\forall w((K_w(p) \wedge K_w(\text{p entails q})) \rightarrow (K_w(\text{q) through inference from p}))$.

To sum up, naïve closure as a general principle is false. There are various other reasons why closure can hold for propositions p, q. However, *particular* versions of naïve closure as well as *contingent* variants can be true.

The Relations between Closure and Inferential Knowledge

Closure is usually regarded as somehow expressing the idea that person can extend their knowledge by deduction from proposition they already know. One way of fleshing out the relation between closure and inferential knowledge is naïve closure. But *general* naïve closure is false. Only *particular* variants of naïve closure are true. What can one conclude from these facts about the relations between knowledge-closure and inferential knowledge? There are various possible reasons, why closure holds. The standard reason that q is known trough inference from p if p is known and (p entails q) is known is only one them. General closure is true iff there is always a reason for knowing q if one knows p and (p entails q). This reason can be the standard reason but any other reason as well. One formulation, which captures the fact that knowing q through inference from p is an important reason but not the only one, is to claim that the standard reason "does the rest": Whenever there is no other reason for knowing q, then q is known through inference from p.

One possible way of expressing this idea is by claiming the following:

- Closure is true iff q is known through inference from p, whenever p and (p entails q) are known, if q is not known for any other reason.

On the one hand, this formulation focuses the attention on the relation between knowledge-closure and inferential knowledge. But, on the other hand, it is trivially true, since it only states that general closure is true iff there is a reason that q is known. Even if one could not know q through inference from p at all, the above claim would be true. Furthermore, it holds for any arbitrary reason as well such as the following:

- Closure is true iff q is known for the reason that the earth is round whenever p and (p entails q) are known), if q is not known for any other reason.

Again, this claim is true because it simply states that general closure is true iff there is a sufficient reason for knowing q if p and (p entails q) are known; either because the earth is round or for any other reason. It is still true, if closure is never true because of the roundness of the earth.

For a non-trivial formulation of the relations between knowledge-closure and inferential knowledge, therefore, we have to explicate the intuition that the standard reason has to do the rest differently. This can be achieved by explicitly listing alternative reasons for closure in the following way:

- If p is not necessarily false,
- if p entails q,
- if it is possible to know that p entails q,
- if q is not necessarily known,
- if q is not necessarily known, if q is true,
- if p cannot only be known through inference from q,
- if q is not known for any other reason such as empirical knowledge, knowledge by testimony, knowledge through inference from other propositions r1, r2, …
- …

then closure is true iff S knows q through inference from p, if S knows p and knows (p entails q).

Another way of putting this is to say that closure is true if the standard reason is fact for p, q and S, if none of the alternative reasons listed above is the case.

If all possible alternative reasons for closure are explicitly listed in a statement like the one above, then it is a non-trivial statement about the relations between knowledge-closure and inferential knowledge. It gives us non-tautological information under what conditions q has to be known through inference from p to make knowledge-closure true for p and q.

One can summarize the role, which inferential knowledge has to play for knowledge-closure by each of the following synonymous claims:

- If no alternative reason for closure is fact, then closure is true if S knows q *through inference* from p, if S knows p and knows that p entails q.

- If no alternative reason for closure is fact and if there are cases such that S knows p and (p entails q) and S still does not know q *through inference* from p, then closure is false.

Naïve closure, which is the claim that q is known *through inference* from p if p and (p entails q) are known, might capture our first intuition about the relation between knowledge-closure and inferential knowledge. However, naïve closure is false. The relations between closure and inferential knowledge are more complex. Closure can be true for various reasons and the standard reason is only one of them. Therefore, closure is true iff the standard reason is always the case if no other reasons is fact.

8. *Variants of Closure*

The presented formulations of the relations between knowledge-closure and inferential knowledge can give us a better understanding of how closure as a general principle is discussed in literature: Some authors like Dretske (1970 and 2005) and Nozick (1981) refute closure for principle reasons and, therefore, they refute any variant of it. Another strategy is to regard closure as too strong and to try to replace it by a weaker and, hence, more adequate principle.

Closure is an implication. Therefore, there are, generally speaking, two strategies of weakening closure: Firstly by strengthening its premises and, secondly, by weakening its consequent.[5] The first strategy is more popular than the second one. One plausible explanation is that the notion, which counts most in epistemology, is knowledge and that, hence, there is stronger interest in sufficient conditions for knowledge than in sufficient conditions for weaker notions. Therefore, I will focus my attention in the following on variants of closure with strengthened antecedents.

One problem of the closure principle is what David and Warfield (2008) call the *belief problem*: The consequent of closure is K(q). Knowing q implies believing q, but neither knowing p nor knowing that p entails q nor knowing both propositions seem to imply believing q. On the contrary, it seems easily possible that a person knows p and knows that p implies q but fails to believe q.[6] The person can simply fail to put 2 plus 2 together, but she can also fail for psychological reason. Here is a macabre example for the second case: John knows that the murder of his mother had a key for her house and that the only person with a key is his brother and he knows that this implies that his brother must be the murder. However, for psychological reasons John simply cannot believe that his brother killed their mother. Considering the belief problem, one can weaken closure by adding the premise that the person S believes q. This weaker version of closure states that if

[5] David and Warfield (2008) make this point explicit.

[6] For a brief discussion of the belief problem, also see Blome-Tillmann (2006).

S knows p and knows that p entails q and believes q, then S knows q. Taking into account the general character of any closure principle with all persons and all propositions as its domain, we can formulate the following principle:

Closure2:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y) \wedge B_w(y)) \rightarrow K_w(y))$

The antecedent of closure2 consists of the antecedent of closure1 plus the additional premise that S believes y. The consequent remains the same. Therefore, closure2 is weaker than closure1. Hence, any sufficient reason for closure is also a sufficient reason for closure 2. However, there is obviously a sufficient reason for closure 2, which is not a reason for closure, namely the fact that S does not believe q. If S necessarily does not believe q, then there is a type-2-reason for closure$2_{(S, p, q)}$. If S does believe q contingently in one possible world w1, then there exists a further aspect of a type-3-reason for closure$2_{(S, p, q)}$.

In order to explicate the relation between inferential knowledge and closure2, one has to add not-believing q to the list of alternative reasons. Hence, closure2 is true iff the standard reason for closure2 is fact, if there are no alternative reason for closure2 which are the alternative reasons for closure plus not-believing q. Again, the standard reason has to do the rest to make closure2 true, but the remaining cases are not the same as in case of closure. The standard reason has to capture the same cases as for closure minus the case that q is not believed. Hence, closure2 is true if the standard reason is fact for p, q and S, if none of the alternative reasons is the case including the case that S does not believe q.

Psychological arguments such as the one based on the belief-problem can also be stressed against closure2. One can assume that inferential knowledge always involves performing a mental process of inference. Accepting this, one can argue that knowing p, knowing that p entails q and believing q is still not sufficient for knowing q through inference, because it does not necessarily include an act of inference. The direct way of reacting to this objection against closure2 is to add as a further premise the proposition that S draws an inference from p to q. The resulting variant of closure is the following:

Closure3:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y) \wedge B_w(y) \wedge (w \text{ infers } y \text{ from } x)) \rightarrow K_w(y))$

One can obtain further variants of closure3 by assuming that S needs to infer q from p *and* (p entails q) or from the conjunction (p and (p entails)). Closure3 is obviously weaker than closure2.

If S does not draw an inference from p to q, then the antecedent of closure3 is false and, therefore, closure3 is true. Not drawing an inference from p to q is, therefore, a sufficient reason for the truth of closure3 but it is not a reason for closure2. Again, closure3 holds if the standard case does the rest, but here the remaining cases are those of closure2 minus the case that S does not draw any inference from p. Closure3 is true if the standard reason is fact whenever there is no alternative reason such as the case that S does not believe q and that S does not draw an inference from q.

Neither closure2 nor closure3 address a problem, which David and Warfield (2008) call the *warrant problem*: Closure, closure2 and closure3 indicate different sufficient conditions for knowing q through inference, but they share the view that the way the belief of q is obtained is not decisive for knowing q. But one might have the intuition that a person who believes q for totally implausible reasons fails to know q, although she knows p and knows that p entails q. This objection can be met by incorporating a condition of belief *acquisition* into the antecedent of closure. Williamson (2000) claims that intuitive closure is the principle that knowing $p_1, \ldots, p_n$, competently deducing q and thereby coming to believe q is in general a way of coming to know q. The intuition that the belief acquisition of q has to be the result of the inference from p and (p entails q) has given rise to formulations of closure, which can be called proper basing closure. This version of closure states that S knows q, if S knows p and knows that p entails q and believes q based on deducing it from p and (p entails q). Taking into account the general character of closure again, we can formulate the following principle:

Closure4:

- $\Box\forall w\forall x\forall y((K_w(x) \wedge K_w(x \text{ entails } y) \wedge (B_w(y)$ based on deduction from x and (x entails y))) $\rightarrow K_w(y))$[7]

Closure3 and Closure4 might seem similar but they differ concerning the structure of deduction and believing. According to closure3, performing a mental process of inference is sufficient to guarantee that knowing p, knowing that p entails q and believing q implies knowing q. Closure3 leaves open, whether the reason for believing q is the process of inference itself or something else. Closure4 in contrast states that sufficient conditions for knowing q are only fulfilled in the first case. From a diachronic view, closure3 allows that one believes q *before* performing an inference. Closure4 excludes this possibility. According to closure4, the temporal order of inference and believing is essential, according to closure3, it is not.

[7] This version of closure is the general formulation of closure6 in David and Warfield (2008). Versions of proper basing closure are most prominently defended by John Hawthorne (2004 and 2005), who ads the further premise that S also has to retain her knowledge that p. David and Warfield present as a further variant of proper basing closure the principle that believing q must solely be based on deduction from p and (p entails q).

As in the cases before, there is a further alternative reason for closure4, which is the fact, that believing q is not based on deduction from p and (p entails q). Therefore, closure4 holds iff the standard reason is always the case, if none of the alternative reasons is fact including the case that S does not believe q based on deduction from p.

To sum up, there are various versions of closure attacked and defended in literature. The alternative versions discussed here are weaker principles than closure, which are obtained by strengthening the antecedent by adding a further premise. Closure can be true for various reasons and the standard reason is only one of them. Closure as a general principle is, hence, true iff the standard reason is always the case if none of the alternative reasons is fact. The same holds for every alternative version of closure. The only difference is that there are different alternative reasons for different versions of closure. The weaker the closure principle is, the more alternative reasons for closure exist.

Summary

I have shown that the closure principle can hold for various different reasons. The standard reason that S knows q through inference from p, if S knows p and knows that p entails q, is only one of them. The naïve view on these relations that q is always known through inference from p, if closure is true, is false. Therefore, the relations between knowledge-closure and inferential knowledge are more complex than one might suspect.

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